UNIT – III
TIME RESPONSE ANALYSIS-II

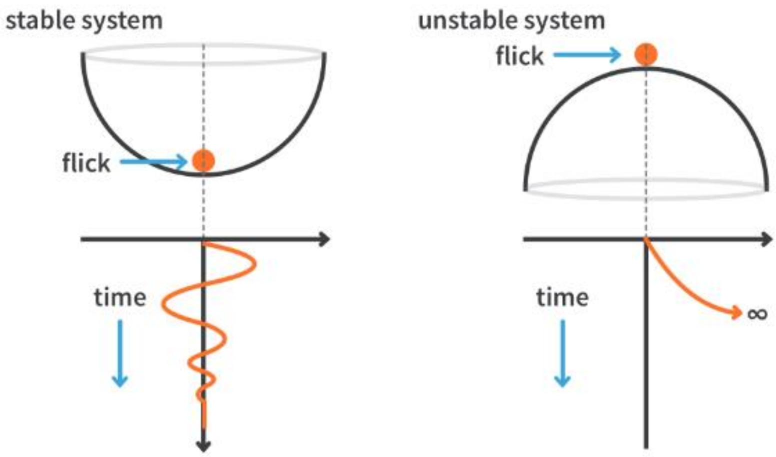
Topics: Concepts of stability, Necessary conditions for Stability, Routh stability criterion, Relative stability analysis- Introduction to Root Locus Technique, Construction of root loci.

CONCEPTS OF STABILITY

A system is said to be stable if its output is under control. Otherwise, it is said to be unstable. A stable system produces a bounded output for a given bounded input.

A system is said to be stable if the system eventually returns to its equilibrium state when the system is subjected to an initial excitation or disturbance.

Example:



Consider a marble and a bowl arranged as shown above. In the first case, the marble is inside the bowl. A small flick will make the marble oscillate about its equilibrium position and it will eventually settle back to its original position. In the second case, the marble is placed on top of an inverted bowl. A small flick in this case would make the marble fall off the bowl and the marble would never come back to its initial position unless you take it and place it back.

[Now, let's come back to the first case where the marble is inside the bowl. What if I flick the marble so hard that it flies out of the bowl? Will the marble come back to its initial position? It's an obvious NO. So, is the system unstable? The system is stable but exciting it with an unbound input makes it difficult to judge if the system is stable or not. This brings us to the concept of the bounded input. An input is said to be bounded if the input lies within definite limits of the system. If it's not bounded, then the input is an unbounded input. As simple as it sounds.]

TYPES OF SYSTEMS BASED ON STABILITY

i) Absolutely Stable System:

If the system is stable for all the range of system component values, then it is known as the absolutely stable system. The open loop control system is absolutely stable if all the poles of the open loop transfer function present in left half of 's' plane. Similarly, the closed loop control system is absolutely stable if all the poles of the closed loop transfer function present in the left half of the 's' plane.

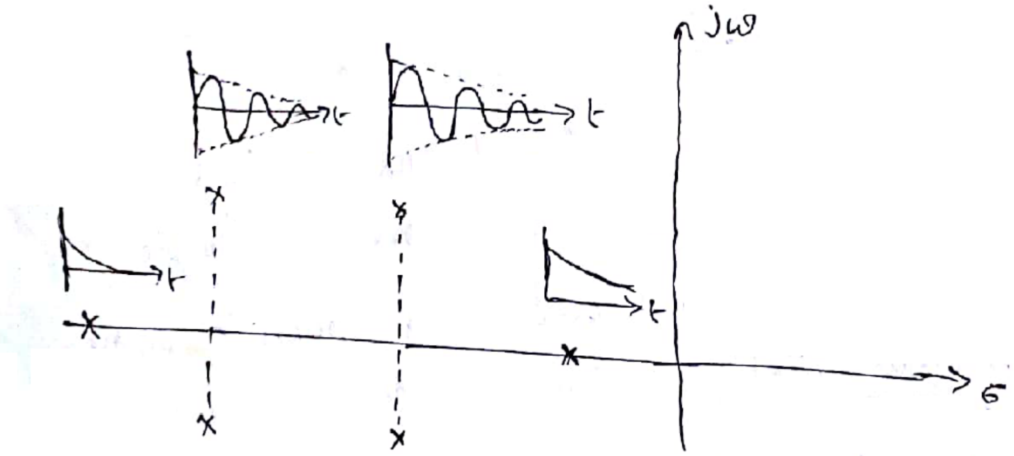
ii) Conditionally Stable System:
If the system is stable for a certain range of system component values, then it is known as conditionally stable system.
iii) Marginally Stable System:
If the system is stable by producing an output signal with constant amplitude and constant frequency of oscillations for bounded input, then it is known as marginally stable system. The open loop control system is marginally stable if any two poles of the open loop transfer function is present on the imaginary axis. Similarly, the closed loop control system is marginally stable if any two poles of the closed loop transfer function is present on the imaginary axis.

RELATIVE STABILITY

The relative stability indicates the looseness of the system to stable region. It is an introduction of the strength or degree of stability.

In time domain the relative stability may be measured by relative settling times of each root or pair of roots. The settling time is inversely proportional to the location of roots of characteristics equation. If the root is located far away from the imaginary axis, then the transients' dies out faster and so the relative stability of the system will improve.

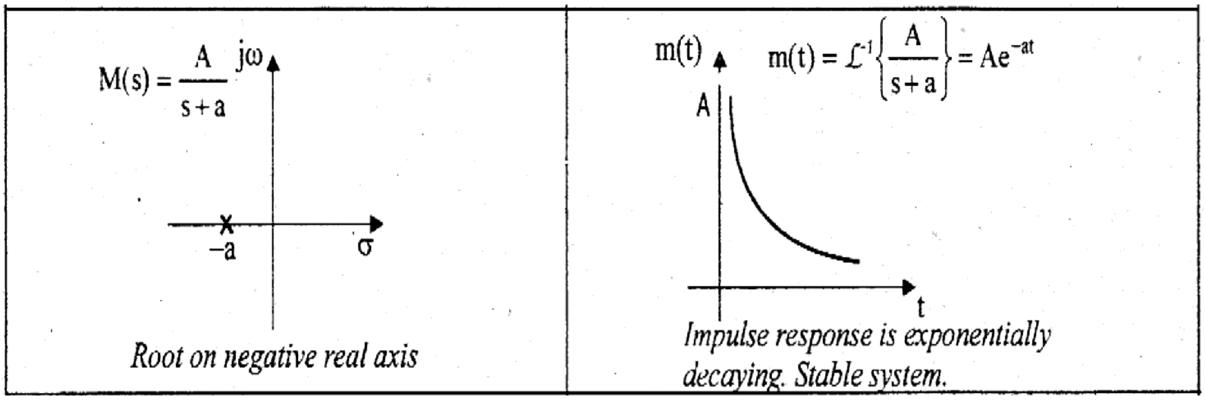
The relative stability for various root locations in s-plane can be shown below.



From the above diagram. As a root (pair of roots) moves farther away from the imaginary axis, the relative stability of the system improves.

NECESSARY CONDITIONS FOR STABILITY (OR)

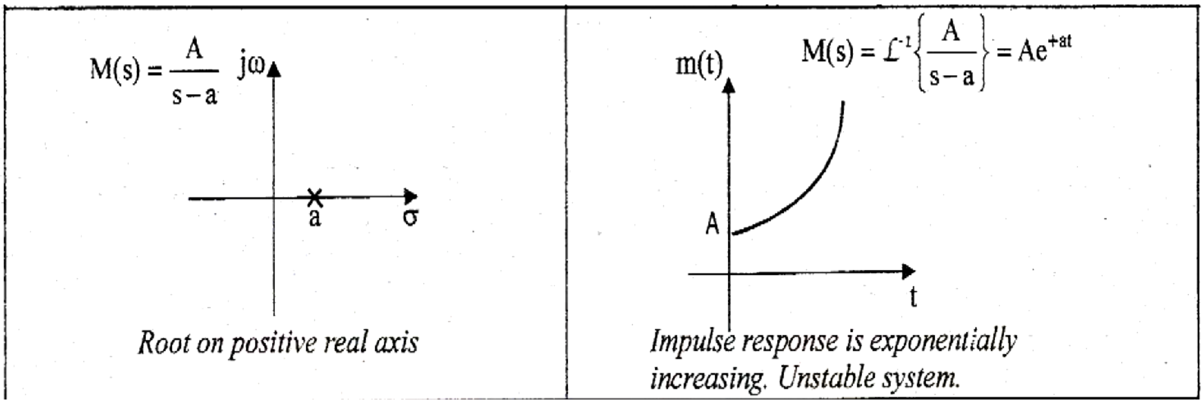
EFFECT OF LOCATION OF POLES ON STABILITY



Root on negative real axis

Impulse response is exponentially decaying. Stable system.

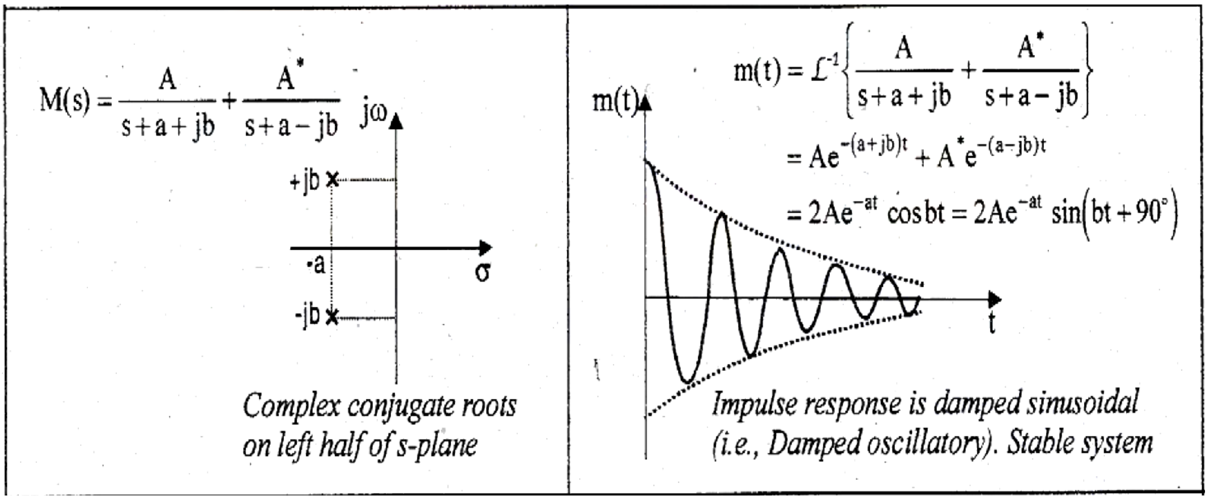
1. If all the roots of characteristic equation have negative real parts (i.e. lying on left half s-plane) then the impulse response is bounded (i.e., it decreases to zero as t tends to ∞). Hence $\int_0^\infty m(\tau)\, d\tau$ is finite and the system is bounded-input bounded output stable.



Root on positive real axis

Impulse response is exponentially increasing. Unstable system.

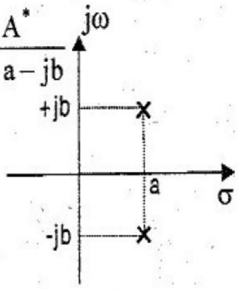
2. If any roots of the characteristic equation has a positive real part (i.e., lying on right half s-plane) then impulse response is unbounded (i.e. increases to ∞ as t tends to ∞). Hence $\int_0^\infty m(\tau)\, d\tau$ is infinite and so system is unstable.



Complex conjugate roots on left half of s-plane

Impulse response is damped sinusoidal (i.e., Damped oscillatory). Stable system

$$M(s) = \frac{A}{s-a+jb} + \frac{A^*}{s-a-jb}$$

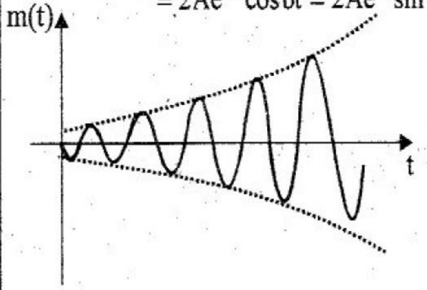


Complex conjugate roots on right half of s-plane

$$m(t) = \mathcal{L}^{-1}\left\{\frac{A}{s-a+jb} + \frac{A^*}{s-a-jb}\right\}$$

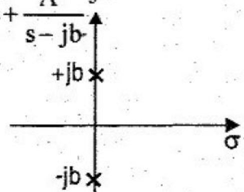
$$= Ae^{-(-a+jb)t} + A^*e^{-(-a-jb)t}$$

$$= 2Ae^{at}\cos bt = 2Ae^{at}\sin(bt + 90^\circ)$$



Impulse response is exponentially increasing sinusoidal (i.e., Amplitude of oscillations exponentially increases with time). Unstable system.

$$M(s) = \frac{A}{s+jb} + \frac{A^*}{s-jb}$$

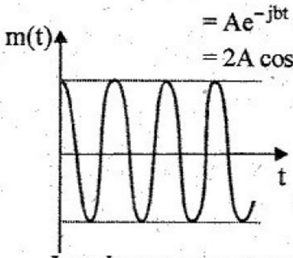


Single pair of roots on imaginary axis

$$m(t) = \mathcal{L}^{-1}\left\{\frac{A}{s+jb} + \frac{A^*}{s-jb}\right\}$$

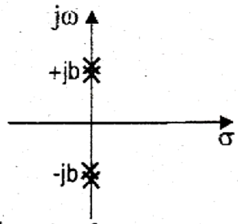
$$= Ae^{-jbt} + A^*e^{+jbt}$$

$$= 2A\cos bt = 2A\sin(bt + 90^\circ)$$



Impulse response is oscillatory
Marginally stable

$$M(s) = \frac{A}{(s+jb)^2} + \frac{A^*}{(s-jb)^2}$$

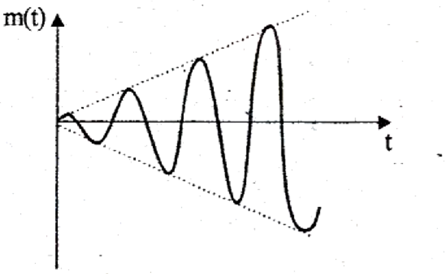


Double pair of roots on imaginary axis

$$m(t) = \mathcal{L}^{-1}\left\{\frac{A}{(s+jb)^2} + \frac{A^*}{(s-jb)^2}\right\}$$

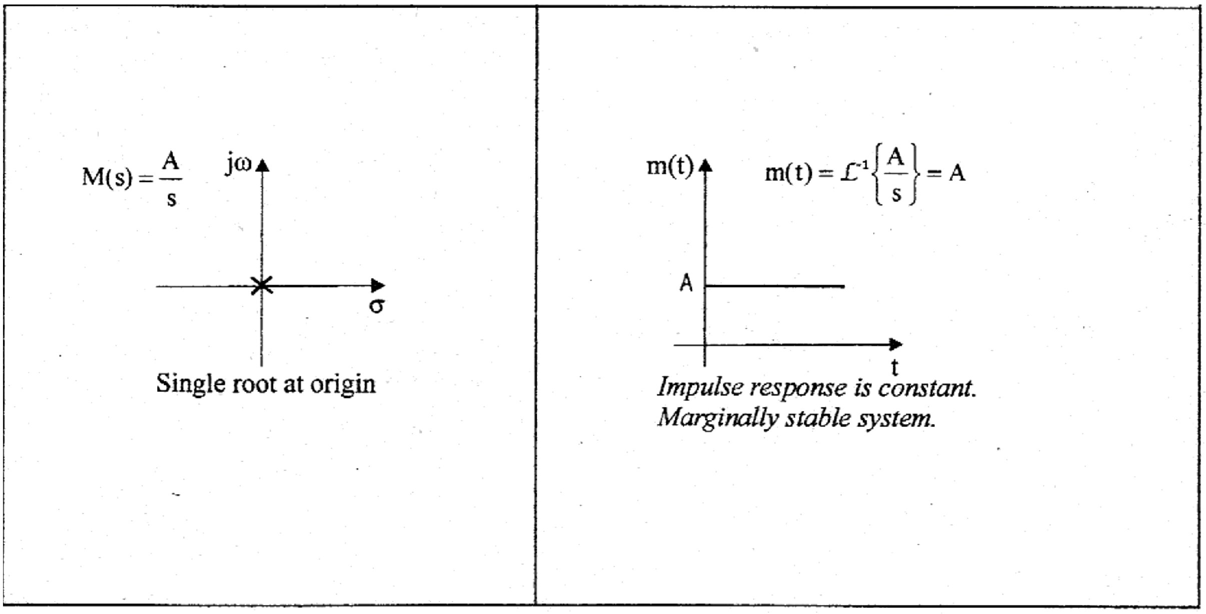
$$= At\,e^{-jbt} + A^*\,t\,e^{+jbt}$$

$$= 2At\cos bt = 2At\sin(bt + 90^\circ)$$

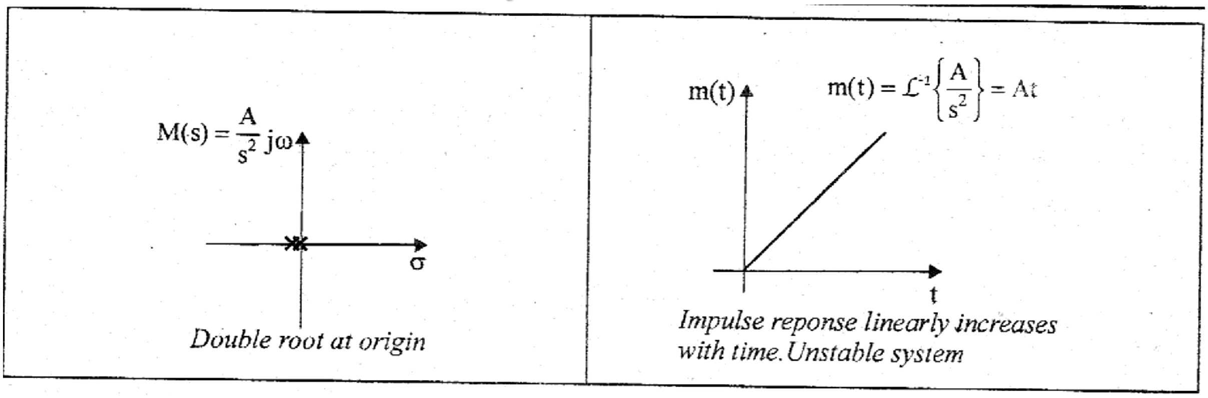


Impulse response is linearly increasing sinusoidal (i.e., amplitude of oscillations linearly increases with time). Unstable system.

3. If the characteristic equation has repeated roots on the imaginary axis then impulse response is unbounded (i.e., it increases to ∞ as t tends to ∞). Hence $\int_0^\infty m(\tau)\,d\tau$ is finite and so system is unstable.



5. If the characteristic equation has single root at origin then the impulse response is bounded (i.e., it has constant amplitude) but $\int_0^\infty m(\tau)\, d\tau$ is infinite and so the system is unstable.



6. If the characteristic equation has repeated roots at origin then the impulse response is unbounded (i.e. it linearly increases to infinity as t tends to ∞) and so the system is unstable.
7. In system with one (or) more repeated roots on imaginary axis (or) with single root at origin, the output is bounded for bounded inputs except for the inputs having poles matching the system poles. These cases may be treated as acceptable (or) non-acceptable. Hence when the system has non repeated poles on imaginary axis (or) single pole at origin, it is referred as limitedly (or) marginally stable system.

In summary the following three points may be stated regarding the stability of the system depending on the location of roots of characteristic equation.

(i) If all the roots of characteristic equation has negative real parts, then the system is stable.

(ii) If any roots of the characteristic equation has a positive real part (or) if there is a repeated root on the imaginary axis then the system is unstable.

(iii) If the condition (i) is satisfied except for the presence of one (or) more non repeated roots on the imaginary axis, then the system is limitedly (or) marginally stable.

(iv) If there are repeated roots on jw-axis, the system is unstable.

(v) In the characteristic equation of the system, if all coefficients are positive and no missing terms, then the system is stable.

ROUTH-HURWITZ (RH) CRITERION

The RH stability criteria is an analytical procedure for determining how many roots of the polynomial lying on left half of s-plane or right half of s-plane of the system.

Statement:

The necessary and sufficient condition for stability is that all the elements in the first column of the Routh array are **positive**. If this condition is not met, the system is unstable and the number of sign changes in the elements of the first column of Routh array is equal to the number of roots of the characteristic equation are lying on right half of s-plane.

Let the characteristic eq.

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots\ldots + a_n s^0 = 0$$

The coeffs of polynomial are arranged as

s^n	a_0	a_2	a_4
s^{n-1}	a_1	a_3	a_5
s^{n-2}	b_1	b_2	b_3
s^{n-3}	b_4	b_5	b_6
$\vdots$			
s^0			

where $b_1 = \dfrac{a_1 a_2 - a_0 a_3}{a_1}$ $\quad b_2 = \dfrac{a_1 a_4 - a_0 a_5}{a_1}$

$b_4 = \dfrac{b_1 a_3 - a_1 b_2}{b_1}$ $\quad b_5 = \dfrac{b_1 a_5 - a_1 b_3}{b_1}$

If all the coe.s of the first column are +ve. then the system is stable.

Example: Check the stability of the system whose characteristic equation is

$$s^4 + 2s^3 + 6s^2 + 4s + 1 = 0$$

SOL:

s^4	1	6	1
s^3	2	4	
s^2	$\frac{6\times2-4}{2}=4$	$\frac{2-0}{2}=1$	
s^1	$\frac{4\times4-2}{4}=3.5$		
s^0	$\frac{3.5-0}{3.5}=1$		

i.e.

s^4	1	6	1
s^3	2	4	0
s^2	4	1	0
s^1	3.5	0	
s^0	1		

All coe.s of 1st column are +ve.

∴ the given system is stable.

SPECIAL CASES

CASE-I: Any one row of Routh table is Zero.

This condition indicates that there are symmetrically located roots in the s-plane.

The Polynomial whose coe.s are the elements of the row just above the row of zeros in the Routh array is called Auxiliary Polynomial. The order of auxiliary polynomial is always even.

For this case, the following procedure can be used.

→ Determine the Auxiliary polynomial A(s)

→ Differentiate the auxiliary polynomial wrt s i.e $\frac{d}{ds}(A(s))$

→ The row of zeros is replaced with coe.s of $\frac{d}{ds}(A(s))$.

→ then continue the construction of array.

Example:
The characteristic equation of the system is

$$S^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$$

Determine the location of roots on s-plane and hence the stability of the system.

SOL:

The Routh array is

s^6	1	8	20	16
s^5	2	12	16	0
s^4	2	12	16	0
s^3	0	0	0	0
s^2				
s^1				
s^0				

∴ Auxiliary eq. $A(S) = 2S^4 + 12S^2 + 16 = 0$

$$\frac{d(A(S))}{dS} = 8S^3 + 24S$$

s^6	1	8	20	16
s^5	2	12	16	0
s^4	2	12	16	0
s^3	8	24	0	0
s^2	6	16	0	0
s^1	2.67	0		
s^0	16			

No sign change in the first column. So the system may be stable.
The roots are lying on imaginary axis because one row is zero.

$\therefore \quad A(s) = 0$

$$2s^4 + 12s^2 + 16 = 0$$

$$s^4 + 6s^2 + 8 = 0$$

$$(s^2+4)\ (s^2+2) = 0$$

$$s = \pm j2 \qquad s = \pm j\sqrt{2}$$

the roots are non repeated on imaginary axis hence the system is marginally stable.

Location of the roots:

The number roots lying on right half of s-plane = 0
The number roots lying on left half of s-plane = 2
The number roots lying on imaginary axis of s-plane = 4

CASE-II : First element of a row is Zero.

Because of this zero, the terms in next row become infinite and Routh's test breaks down.

To overcome this difficulty the following methods are used.

Method : 1 → (i) Replace zero by ϵ (small no.) and complete the array with ϵ.

(ii) Examine the sign change by taking $\epsilon \to 0$.

Method : 2 → put $s = \frac{1}{z}$ and Apply the Routh's test on the modified eq. in terms of z.

Example:
Examine the stability of the characteristic equation

$$s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$$

SOL:

method : 1

s^5	1	2	3
s^4	1	2	5
s^3	ε	-2	0
s^2	$\frac{2\varepsilon+2}{\varepsilon}$	5	0
s^1	$\frac{-4\varepsilon-4-5\varepsilon^2}{2\varepsilon+2}$	0	
s^0	5		

Apply $\varepsilon \to 0$

Then the routh array is

s^5	1	2	3
s^4	1	2	5
s^3	0	-2	0
s^2	∞	5	0
s^1	-2	0	0
s^0	5	0	0

method : 2

Replace $s = \frac{1}{z}$

$$\Rightarrow \frac{1}{z^5} + \frac{1}{z^4} + 2\frac{1}{z^3} + 2\frac{1}{z^2} + 3\frac{1}{z} + 5 = 0$$

$$5z^5 + 3z^4 + 2z^3 + 2z^2 + z + 1 = 0$$

The Routh array is

z^5	5	2	1
z^4	3	2	
z^3	$-4/3$	$-2/3$	0
z^2	$1/2$	$+1$	
z^1	2	0	
z^0	1	0	

∴ There are two sign changes i.e. the system having two poles in the right half of s-plane. Hence the system is unstable.

PROBLEMS

1) For the system has the characteristic equation $s^3 + 3s^2 + s + 3 = 0$ Determine its stability.

SOL:

$$(s^2+1)(s+3) = 0$$

$$s^3 + 3s^2 + s + 3 = 0$$

The Routh array is

s^3	1	1
s^2	3	3
s^1	ε	0
s^0	3	

∴ If $\varepsilon \to 0$, all the elements of first column are +ve. ∴ The system is stable system.

2) Determine the range of K for stability of unity feedback system whose open loop transfer function is

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

SOL:

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{K}{s(s+1)(s+2)}}{1+\frac{K}{s(s+1)(s+2)}} = \frac{K}{s(s+1)(s+2)+K}$

The characteristic equation is, $s(s+1)(s+2)+K=0$

$\therefore s(s^2+3s+2)+K=0 \quad \Rightarrow \quad s^3+3s^2+2s+K=0$

The routh array is constructed as shown below.

s^3	:	1	2
s^2	:	3	K
s^1	:	$\frac{6-K}{3}$	
s^0	:	K	

Column-1

$s^1 : \frac{3\times 2 - K\times 1}{3}$

$s^1 : \frac{6-K}{3}$

$s^0 : \frac{\frac{6-K}{3}\times K - 0\times 3}{(6-K)/3}$

$s^0 : K$

For the system to be stable there should not be any sign change in the elements of first column. Hence choose the value of K so that the first column elements are positive.

From s^0 row, for the system to be stable, $K > 0$

From s^1 row, for the system to be stable, $\frac{6-K}{3} > 0$

For $\frac{6-K}{3} > 0$, the value of K should be less than 6.

$\therefore$ The range of K for the system to be stable is $0<K<6$.

3) The open loop transfer function of a unity feedback control system is given by

$$G(s) = \frac{K}{(s+2)(s+4)(s^2+6s+25)}$$

By applying routh criteria, discuss the stability of closed loop system as a function of 'K'. Determine the value of 'K' which will cause sustained oscillations in the closed-loop system. What are the corresponding oscillating frequencies?

SOL:

The closed loop transfer function $\Big\}$ $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{K}{(s+2)(s+4)(s^2+6s+25)}}{1+\frac{K}{(s+2)(s+4)(s^2+6s+25)}} = \frac{K}{(s+2)(s+4)(s^2+6s+25)+K}$

The characteristic equation is given by the denominator polynomial of closed loop transfer function.

The characteristic equation is, $(s+2)(s+4)(s^2+6s+25)+K=0$.

$\therefore (s^2+6s+8)(s^2+6s+25)+K=0 \quad \Rightarrow \quad s^4+12s^3+69s^2+198s+200+K=0$

The routh array is constructed as shown below. The highest power of s in the characteristic equation is even number. Hence form the first row using the coefficients of even powers of s and form the second row using the coefficients of odd powers of s.

s^4	:	1	69	200+K	..... Row-1
s^3	:	12	198		..... Row-2

Divide s^3 row by 12 to simplify the calculations

s^4	:	1	69	200+K	..... Row-1
s^3	:	1	16.5		..... Row-2
s^2	:	52.5	200+K		..... Row-3
s^1	:	$\frac{666.25-K}{52.5}$			..... Row-4
s^0	:	200+K			..... Row-5

(Column-1)

$$s^2 : \frac{1\times 69 - 16.5\times 1}{1} \quad \frac{1\times(200+K)}{1}$$

$$s^2 : 52.5 \qquad 200+K$$

$$s^1 : \frac{52.5\times 16.5 - (200+K)\times 1}{52.5}$$

$$s^1 : \frac{666.25-K}{52.5}$$

$$s^0 : \frac{\frac{666.25-K}{52.5}\times(200+K)}{(666.25-K)/52.5}$$

$$s^0 : 200+K$$

From s^1 row, for the system to be stable, (666.25-K) > 0.

Since (666.25-K) > 0, should be less than 666.25.

From s^0 row, for the system to be stable, (200+K) > 0

Since (200+K) > 0, K should be greater than -200, but practical values of K starts from 0. Hence K should be greater than zero.

∴ The range of K for the system to be stable is $0<K<666.25$.

When K= 666.25 the s^1 row becomes zero, which indicates the possibility of roots on imaginary axis. A system will oscillate if it has roots on imaginary axis and no roots on right half of s-plane.

When K=666.25, the coefficients of auxiliary equation are given by the s^2 row.

∴ The auxiliary equation is, $52.5s^2+200+K=0$

$$52.5s^2 + 200 + 666.25 = 0$$

$$s^2 = \frac{-200-666.25}{52.5} = -16.5$$

$$s = \pm\sqrt{-16.5} = \pm j\sqrt{16.5} = \pm j4.06$$

When K = 666.25, the system has roots on imaginary axis and so it oscillates. The frequency of oscillation is given by the value of root on imaginary axis.

∴ The frequency of oscillation, $\omega = 4.06$ rad/sec.

4) For a unity feedback system,

$$G(s) = \frac{K}{s(1+0.4s)(1+0.25s)}$$

Find the range of values of 'K', marginal value of 'K' and frequency of sustained oscillations.

SOL:

Characteristic equation, 1 + G(s)H(s) = 0 and H(s) = 1

$$1 + \frac{K}{s(1+0.4s)(1+0.25s)} = 0$$

$$s[1 + 0.65s + 0.1s^2] + K = 0$$

$$0.1s^3 + 0.65s^2 + s + K = 0$$

s^3	0.1	1	From s^0, $K > 0$
s^2	0.65	K	From s^1,
s^1	$\frac{0.65 - 0.1K}{0.65}$	0	$0.65 - 0.1K > 0$ $\therefore\ 0.65 > 0.1K$
s^0	K		$\therefore\ 6.5 > K$

$\therefore$ Range of values of K, $0 < K < 6.5$.

The marginal value of `K' is a value which makes any row other than s^0 as row of zeros.

$$\therefore \quad 0.65 - 0.1\,K_{mar} = 0$$

$$\therefore \quad \boxed{K_{mar} = 6.5}$$

To find frequency, find out roots of auxiliary equation at marginal value of `K'.

$$A(s) = 0.65s^2 + K = 0\ ;$$

$$\therefore \quad 0.65s^2 + 6.5 = 0 \qquad \because\ K_{mar} = 6.5$$

$$s^2 = -10$$

$$s = \pm\, \mathbf{j\ 3.162}$$

Comparing with $s = \pm j\omega$

ω = Frequency of oscillations

= **3.162 rad/sec.**

5) Find the range of 'K' for the system to be stable, if

$$G(s)H(s) = \frac{K(1+s)^2}{s^3}$$

SOL:

Characteristic equation :

$$1 + G(s)H(s) = 0$$

$$1 + 1 + \frac{K(1+s)^2}{s^3} = 0$$

$$s^3 + K\,[s^2 + 2s + 1] = 0$$

$$s^3 + Ks^2 + 2Ks + K = 0$$

s^3	1	2K	from s^0 and s^2, $K > 0$
s^2	K	K	$\frac{2K^2 - K}{K} > 0$
s^1	$\frac{2K^2 - K}{K}$	0	$2K - 1 > 0$
s^0	K		$2K > 1 \quad \therefore\ K > \frac{1}{2}$

Range of values of K is $K > 0.5$ and $< \infty$

6) Determine the number of roots on imaginary axis for the characteristic equation

$$s^5 + 6s^4 + 15s^3 + 30s^2 + 44s + 24 = 0$$

SOL:

Routh's array

s^5	1	15	44
s^4	6	30	24
s^3	10	40	0
s^2	6	24	0
s^1	0	0	0 ⇐ Special case
s^0			

Since we have a row of all zeros we take the row above this take it as auxiliary equation and differentiate.

$$6s^2 + 24 = 0 \quad \text{Auxiliary equation.}$$

differentiating $\quad 12s = 0$

Now replace row of zeros with coefficient of the above equation.

s^5	1	15	44
s^4	6	30	24
s^3	10	40	0
s^2	6	24	0
s^1	12	0	0
s^0	24		

There is no sign change i.e. no root is in right half.

Now roots can be found out taking roots of auxiliary equation.

$$6s^2 + 24 = 0$$

$$\therefore \quad s^2 + 4 = 0$$

$$s^2 = -4$$

$$\therefore \quad s = \pm 2j$$

Two roots on imaginary axis at $s = +2j$ and $-2j$

7) The open loop transfer function of a unity feedback control system is given by

$$G(s) = \frac{Ke^{-s}}{s(s^2 + 5s + 9)}$$

Determine the maximum value of 'K' for the stability of the closed loop system.

SOL:

$$\therefore G(s) = \frac{Ke^{-s}}{s(s^2+5s+9)} \approx \frac{K(1-s)}{s(s^2+5s+9)}$$

$$\left.\begin{matrix}\text{The closed loop}\\ \text{transfer function}\end{matrix}\right\} \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{K(1-s)}{s(s^2+5s+9)}}{1+\frac{K(1-s)}{s(s^2+5s+9)}} = \frac{K(1-s)}{s(s^2+5s+9)+K(1-s)}$$

The characteristic equation is given by the denominator polynomial of closed loop transfer function.

$\therefore$ The characteristic equation is, $s(s^2+5s+9)+K(1-s)=0$

$\therefore s(s^2+5s+9)+K(1-s) = s^3+5s^2+9s+K-Ks = 0 \Rightarrow s^3+5s^2+(9-K)s+K=0$

The routh array of characteristic polynomial is constructed as shown below.

s^3	:	1	9 – K
s^2	:	5	K
s^1	:	9 – 1.2K	
s^0	:	K	

From s^1 row, for stability of the system, $(9-1.2K)>0$

If $(9-1.2K)>0$ then $1.2K<9$; $\therefore K < \frac{9}{12} = 7.5$

Finally we can conclude that for stability of the system K should be in the range of $0<K<7.5$

ROOT LOCUS

The Root locus technique is a powerful tool for adjusting the location of closed loop poles to achieve the desired system performance by varying one/more system parameters.

The roots of the characteristic eq. depend on the open loop gain K. When $K=0$, the roots are given by open loop poles and when $K=\infty$, the roots are given by open loop zeros.

The path taken by the root of the characteristic eq. when open loop gain K is varied from $0-\infty$ is called Root locus / Root loci.

It is a graphical method of plotting the locus of roots in s-plane as a given system parameter is varied from zero to ∞. The root locus also provides a measure of sensitivity of roots to the variation in the parameter being considered.

Magnitude & Angle Condition

The characteristic eq. $1+G(s)H(s)=0$

$$G(s)H(s)=-1$$

Magnitude condition is $|G(s)H(s)|=1$

Angle " $\angle G(s)H(s)=\pm 180$

In general : $\angle G(s)H(s)=\pm(2q+1)180$

where $q=0,1,2\ldots\ldots$

(i.e. odd multiple of 180).

RULES FOR CONSTRUCTING ROOT LOCUS

Rule : 1 The Root locus is symmetrical about real axis.

Rule : 2 No. of branches, $N = P$ if $P > Z$

$= Z$ if $Z > P$ (Rarely)

Rule : 3 Each branch of the root locus originates from an open-loop pole at $K=0$ and terminates at open loop zero at $K = \infty$.

The no. of branches of root locus terminating on infinity is equal to $(P-Z)$.

Rule : 4 A point on the real axis will lie on the root locus if only the sum of open loop poles & zero is odd. to the right of the point.

Rule : 5 Asymptotes :- These are the straight lines (Dotted lines) which are parallel to root locus (branches) going to infinity and meet the root locus at ∞.

These asymptotes making angles with real axis.

The angle of asymptotes

$$\phi_A = \frac{\pm 180(2q+1)}{P-Z} \qquad q = 0, 1, 2, \ldots (P-Z)-1$$

Rule : 6 The point of intersection of the asymptotes with the real axis is called centroid.

$$\text{centroid } \sigma_A = \frac{\Sigma \text{ Poles} - \Sigma \text{ Zeros}}{P-Z}$$

Rule : 7 Break-away & Break in Points

Break-away point is defined as the point at which the root locus comes out of the real axis.

Break in point is defined as a point at which root locus enters the real axis.

These points are determined by $\frac{dk}{ds} = 0$ & find the value of s.

If n branches of root locus meet a point, then they break away at an angle of $\pm \frac{180}{n}$.

Rule : 8 The angle of departure of the root locus from a complex pole is given by

$$\phi_d = 180 - (\phi_p - \phi_z)$$

where ϕ_p = sum of all angles subtended by remaining poles.

ϕ_z = " " " zeros.

The angle of departure is tangent to the root locus at the complex pole.

Rule : 9 The angle of arrival of the root locus at complex zero is given by

$$\phi_a = 180 - (\phi_z - \phi_p)$$

ϕ_z = sum of all the angles subtended by remaining zeros.

ϕ_p = " " poles.

The angle of arrival is tangent to the root locus at the complex zero.

Rule : 10 The intersection (ω) of root locus branches with the imaginary axis can be determined by use of RH criterion or by letting $s = j\omega$ in the characteristic eq. and equating the real part & imaginary part to zero to solve for ω & K.

Rule 11 : The open-loop gain K at any point $s = s_a$ on the root locus is given by,

$$K = \frac{\prod_{i=1}^{n} |s_a + p_i|}{\prod_{i=1}^{m} |s_a + z_i|} = \frac{\text{Product of vector lengths from open loop poles to the point } s_a}{\text{Product of vector lengths from open loop zeros to the point } s_a}$$

PROCEDURE FOR CONSTRUCTING ROOT LOCUS

Step 1 : Locate the poles and zeros of G(s)H(s) on the s-plane. The root locus branch starts from open loop poles and terminates at zeros.

Step 2 : Determine the root locus on real axis.

Step 3 : Determine the asymptotes of root locus branches and meeting point of asymptotes with real axis.

Step 4 : Find the breakaway and breakin points.

Step 5 : If there is a complex pole then determine the angle of departure from the complex pole. If there is a complex zero then determine the angle of arrival at the complex zero.

Step 6 : Find the points where the root loci may cross the imaginary axis.

Step 7 : Take a series of test points in the broad neighbourhood of the origin of the s-plane and adjust the test point to satisfy angle criterion. Sketch the root locus by joining the test points by smooth curve.

Step 8 : The value of gain K at any point on the locus can be determined from magnitude condition.

The value of K at a point $s = s_a$, is given by,

$$K = \frac{\text{product of length of vectors from poles to the point, } s = s_a}{\text{product of length of vectors from finite zeros to the point, } s = s_a}$$

Effects of Addition of Poles

1) There is a change in the shape of the root locus.
2) The root locus shifts towards the imaginary axis. i.e. toward right hand side
3) The system becomes oscillatory.
4) The angles of asymptotes are decreased and the value of K is decreased.
5) Gain margin and relative stability decreases.
6) There is a reduction in the range of K.
7) Settling time is increased.
8) A sluggish response can be changed to a quicker response.

Effects of Addition of Zeros

1) There is a change in shape of root locus.
2) The root locus shifts towards the left hand side.
3) System stability increases.
4) The Range of K increases.
5) The settling time is increased.

PROBLEMS

1) A unity feedback control system has an open loop transfer function

$$G(s) = \frac{K}{s(s^2 + 4s + 13)}$$

Sketch the root locus.

SOL:

Step 1 : To locate poles and zeros

The poles of open loop transfer function are the roots of the equation, $s(s^2+4s+13)=0$.

The roots of the quadratic are, $s = \frac{-4 \pm \sqrt{4^2 - 4 \times 13}}{2} = -2 \pm j3$

$\therefore$ The poles are lying at $s = 0, -2 + j3$ and $-2 - j3$.

Let us denote the poles as P_1, P_2, and P_3.

Here, $P_1 = 0$, $P_2 = -2 + j3$ and $P_3 = -2 - j3$.

The poles are marked by X (cross) as shown in fig

Step 2 : To find the root locus on real axis

There is only one pole on real axis at the origin. Hence if we choose any test point on the negative real axis then to the right of that point the total number of real poles and zeros is one, which is an odd number. Hence the entire negative real axis will be part of root locus. The root locus on real axis is shown as a bold line in fig

Step 3 : To find angles of asymptotes and centroid

Since there are 3 poles, the number of root locus branches are three. There is no finite zero. Hence all the three root locus branches ends at zeros at infinity. The number of asymptotes required are three.

$$\text{Angles of asymptotes} = \frac{\pm 180^{\circ}\,(2q+1)}{n-m} \quad ; \quad q = 0, 1, \ldots\ldots\ldots n-m$$

Here n = 3, and m = 0. $\therefore$ q = 0, 1, 2, 3 .

When q = 0, $\text{Angles} = \pm\frac{180^{\circ}}{3} = \pm 60^{\circ}$

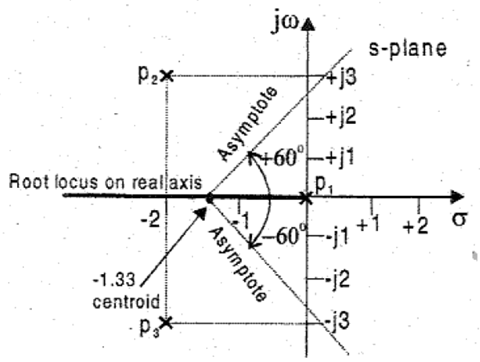
When q = 1, $\text{Angles} = \pm\frac{180^{\circ}\times 3}{3} = \pm 180^{\circ}$

When q = 2, $\text{Angles} = \pm\frac{180^{\circ}\times 5}{3} = \pm 300^{\circ} = \mp 60^{\circ}$

When q = 3, $\text{Angles} = \pm\frac{180^{\circ}\times 7}{3} = \pm 420^{\circ} = \pm 60^{\circ}$

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m} = \frac{0-2+j3-2-j3-0}{3} = \frac{-4}{3} = -1.33$$

The centroid is marked on real axis and from the centroid the angles of asymptotes are marked using a protractor. The asymptotes are drawn as dotted lines as shown in fig



Step 4 : To find the breakaway and breakin points

$$\left.\begin{array}{l}\text{The closed loop}\\ \text{transfer function}\end{array}\right\} \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{K}{s\,(s^2+4s+13)}}{1+\dfrac{K}{s\,(s^2+4s+13)}} = \frac{K}{s\,(s^2+4s+13)+K}$$

The characteristic equation is, $s(s^2+4s+13)+K=0$

$$\therefore\ s^3+4s^2+13s+K=0 \ \Rightarrow\ K=-s^3-4s^2-13s$$

On differentiating the equation of K with respect to s we get,

$$\frac{dK}{ds} = -(3s^2+8s+13)$$

Put $\frac{dK}{ds} = 0$

$$\therefore\ -(3s^2+8s+13)=0 \quad \Rightarrow \quad (3s^2+8s+13)=0$$

$$\therefore s = \frac{-8 \pm \sqrt{8^2 - 4 \times 13 \times 3}}{2 \times 3} = -1.33 \pm j1.6$$

Check for K : When, $s = -1.33 + j1.6$, the value of K is given by,

$$K = -(s^3 + 4s^2 + 13s) = -[(-1.33 + j1.6)^3 + 4(-1.33 + j1.6)^2 + 13(-1.33 + j1.6)]$$

$$\neq \text{positive and real.}$$

Also it can be shown that when $s = -1.33 - j1.6$ the value of K is not equal to real and positive.

Since the values of K for, $s = -1.33 \pm j1.6$, are not real and positive, these points are not an actual breakaway or breakin points. The root locus has neither breakaway nor breakin point.

Step 5 : To find the angle of departure

Let us consider the complex pole p_2 shown in fig . Draw vectors from all other poles to the pole p_2 as shown in fig Let the angles of these vectors be θ_1 and θ_2.

Here, $\theta_1 = 180° - \tan^{-1}(3/2) = 123.7°$; $\theta_2 = 90°$

Angle of departure from the complex pole $p_2 = 180° - (\theta_1 + \theta_2)$

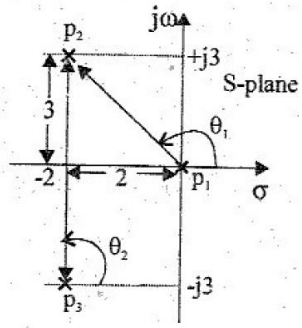
$$= 180° - (123.7° + 90°)$$

$$= -33.7°$$

The angle of departure at complex pole p_3 is negative of the angle of departure at complex pole A.

$\therefore$ Angle of departure at pole $p_3 = +33.7°$

Mark the angles of departure at complex poles using protractor.



Step 6 : To find the crossing point on imaginary axis

The characteristic equation is given by,

$$s^3 + 4s^2 + 13s + K = 0$$

Put $s = j\omega$

$$(j\omega)^3 + 4(j\omega)^2 + 13(j\omega) + K = 0 \quad \Rightarrow -j\omega^3 - 4\omega^2 + 13j\omega + K = 0$$

On equating imaginary part to zero, we get,

$$-\omega^3 + 13\omega = 0$$

$$-\omega^3 = -13\omega$$

$$\omega^2 = 13 \Rightarrow \omega = \pm\sqrt{13} = \pm 3.6$$

On equating real part to zero, we get,

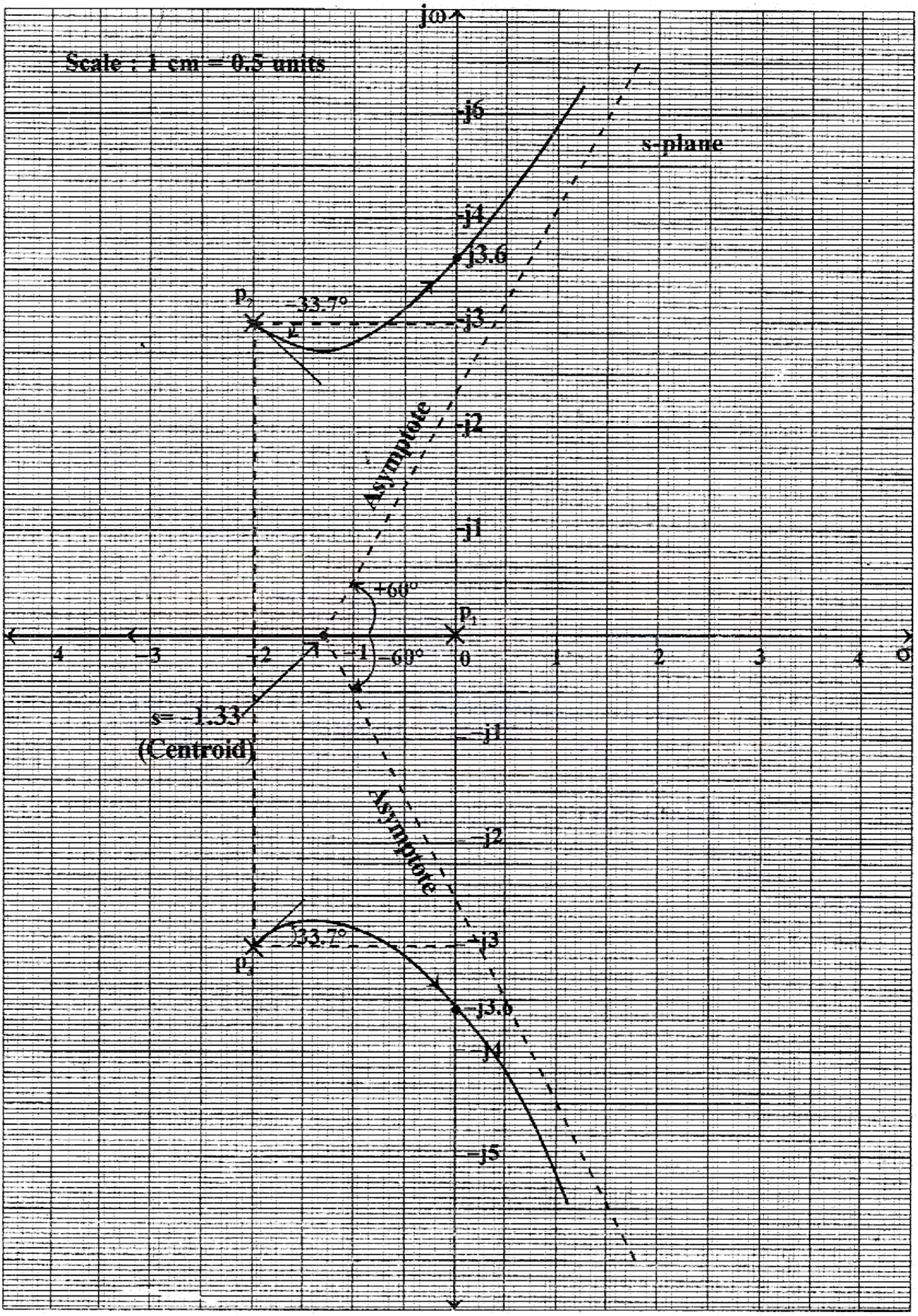
$$-4\omega^2 + K = 0$$

$$K = 4\omega^2$$

$$= 4 \times 13 = 52$$

The crossing point of root locus is $\pm$ j3.6. The value of K at this crossing point is K = 52. (This is the limiting value of K for the stability of the system).

The complete root locus sketch is shown in fig The root locus has three branches one branch starts at the pole at origin and travel through negative real axis to meet the zero at infinity. The other two root locus branches starts at complex poles (along the angle of departure), crosses the imaginary axis at $\pm$j3.6 and travel parallel to asymptotes to meet the zeros at infinity.


Scale : 1 cm = 0.5 units
jω
s-plane
j6
j4
j3.6
j3
j2
j1
P₂
−33.7°
Asymptote
+60°
−60°
P₁
s= −1.33
(Centroid)
−j1
−j2
−j3
−j3.6
−j4
−j5
33.7°
P₃
σ
4
−3
−2
−1
−1
0
1
2
3
4

2) Sketch the root locus of the closed loop system whose open loop transfer function is

$$G(s)H(s) = \frac{K}{s(s+4)(s+6)}$$

SOL:

Poles are at $s = 0,\ -4$ and -6. Hence, $n = 3$. No zero. Therefore, $m = 0$. Parts of the real axis that will be on the root locus are $-4 < s < 0$ and $-\infty < s < -6$.

Asymptotes: Number $= n - m = 3$. Therefore, $q = 0,\ 1,\ 2$. The angles are $60°,\ 180°,\ 300°$.

Centroid:

$$\chi_c = \frac{-4-6}{3} = -3.33$$

Breakaway point: It should exist in the interval $-4 < s < 0$. The characteristic equation is

$$s(s+4)(s+6) + K = 0$$

$$s^3 + 10s^2 + 24s + K = 0 \qquad \text{(i)}$$

$$\frac{dK}{ds} = -(3s^2 + 20s + 24) = 0$$

Therefore, $s = -1.57,\ -5.097$

The former point is acceptable, but the latter is not, because it does not lie on the root locus.

Intersection on the $j\omega$-axis: The Routh array is constructed from Eq. (i) as

s^3	1	24
s^2	10	K
s^1	$24 - 0.1K$	0
s^0	K	

For marginal stability,

$$24 - 0.1K_{\text{mar}} = 0$$

or

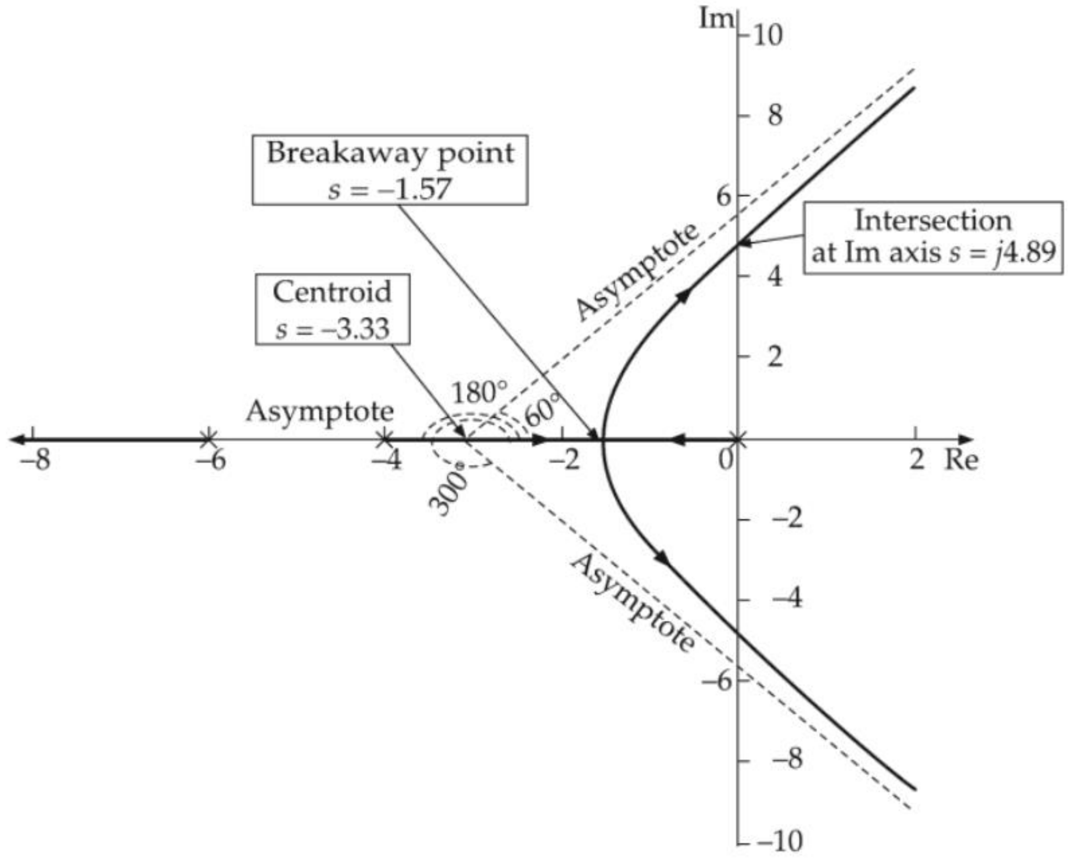
$$K_{\text{mar}} = 240$$

Therefore, from the auxiliary equation, we get

$$10s^2 + 240 = 0$$

Therefore, $s = \pm j4.899$

Having obtained these data, we plot the root locus of Figure



3) Sketch the root locus for 0 < K < ∞ for the system with the open-loop transfer function

$$G(s)H(s) = \frac{K}{s(s+2)(s^2+2s+2)}$$

SOL:

For the given open-loop transfer function $G(s)H(s)$:

The open-loop poles are at $s = 0$, $s = -2$, $s = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm j1$. Therefore, $n = 4$.

There are no finite open-loop zeros. Therefore, $m = 0$.

So the number of branches of root locus = $n = 4$ and the number of asymptotes = $n - m = 4 - 0 = 4$.

The complete root locus is drawn as shown in Figure , as per the rules given as follows:

1. Since the pole-zero configuration is symmetrical with respect to the real axis, the root locus will be symmetrical with respect to the real axis.
2. The four branches of the root locus originate at the open-loop poles $s = 0$, $s = -2$, $s = -1 + j1$ and $s = -1 - j1$, where $K = 0$ and terminate at the open-loop zeros at infinity, where $K = \infty$.
3. There are four asymptotes and the angles of the asymptotes are

$$\theta_q = \frac{(2q+1)\pi}{n-m}, \quad q = 0,1,2,3$$

i.e. $\theta_0 = \frac{\pi}{4}, \quad \theta_1 = \frac{3\pi}{4}, \quad \theta_2 = \frac{5\pi}{4}, \quad \theta_3 = \frac{7\pi}{4}$

4. The point of intersection of the asymptotes on the real axis (centroid) is given by

$$-\sigma = \frac{\text{sum of real parts of poles} - \text{sum of real parts of zeros}}{\text{number of poles} - \text{number of zeros}} = \frac{(0-2-1-1)-(0)}{4-0} = -1$$

5. The root locus exists on the real axis from $s = 0$ to $s = -2$.

6. The break points are given by the solution of the equation $\frac{dK}{ds} = 0$.

$$|G(s)H(s)| = \left|\frac{K}{s(s+2)(s^2+2s+2)}\right| = 1$$

$\therefore$ $$K = s(s+2)(s^2+2s+2)$$

So, $$\frac{d}{ds}[s(s+2)(s^2+2s+2)]$$

i.e. $$\frac{d}{ds}(s^4+4s^3+6s^2+4s) = 0$$

i.e. $$4s^3 + 12s^2 + 12s + 4 = 0$$

i.e. $$s^3 + 3s^2 + 3s + 1 = 0$$

i.e. $$(s+1)^3 = 0$$

Therefore, the break points are at $s = -1$, $s = -1$ and $s = -1$. All are the actual break points.
The break angles at $s = -1$ are

$$\pm\frac{\pi}{r} = \pm\frac{180°}{4} = \pm 45°$$

7. The angle of departure from the complex pole at $s = -1 + j1$ is

$$\theta_d = (2q+1)\pi + \phi$$

where $\phi = -(\theta_1 + \theta_2 + \theta_3) = -(135° + 90° + 45°) = -270°$

$\therefore$ $$\theta_d = \pi - 270° = -90°$$

Hence the angle of departure from the complex pole at $s = -1 - j1$ is $\theta_d = +90°$.

8. The point of intersection of the root locus with the imaginary axis, and the critical value of K are obtained using the Routh criterion. The characteristic equation is

$$1 + G(s)H(s) = 0$$

i.e. $$1 + \frac{K}{s(s+2)(s^2+2s+2)} = 0$$

i.e. $$s^4 + 4s^3 + 6s^2 + 4s + K = 0$$

The Routh table is as follows:

s^4	1	6	K
s^3	4	4	
s^2	5	K	
s^1	$\frac{20-4K}{5}$	0	
s^0	K		

For stability, all the elements in the first column of the Routh array must be positive. Therefore,

$$K > 0$$

and

$$\frac{20-4K}{5} > 0$$

i.e.

$$K < 5$$

Therefore, the range of values of K for stability is $0 < K < 5$. The marginal value of K for stability is $K_m = 5$.

The point of intersection of the root locus with the imaginary axis (i.e. the frequency of sustained oscillations) is given by the solution of the auxiliary equation

$$5s^2 + K = 0$$

i.e.

$$5s^2 + K_m = 0$$

i.e.

$$5s^2 + 5 = 0$$

i.e.

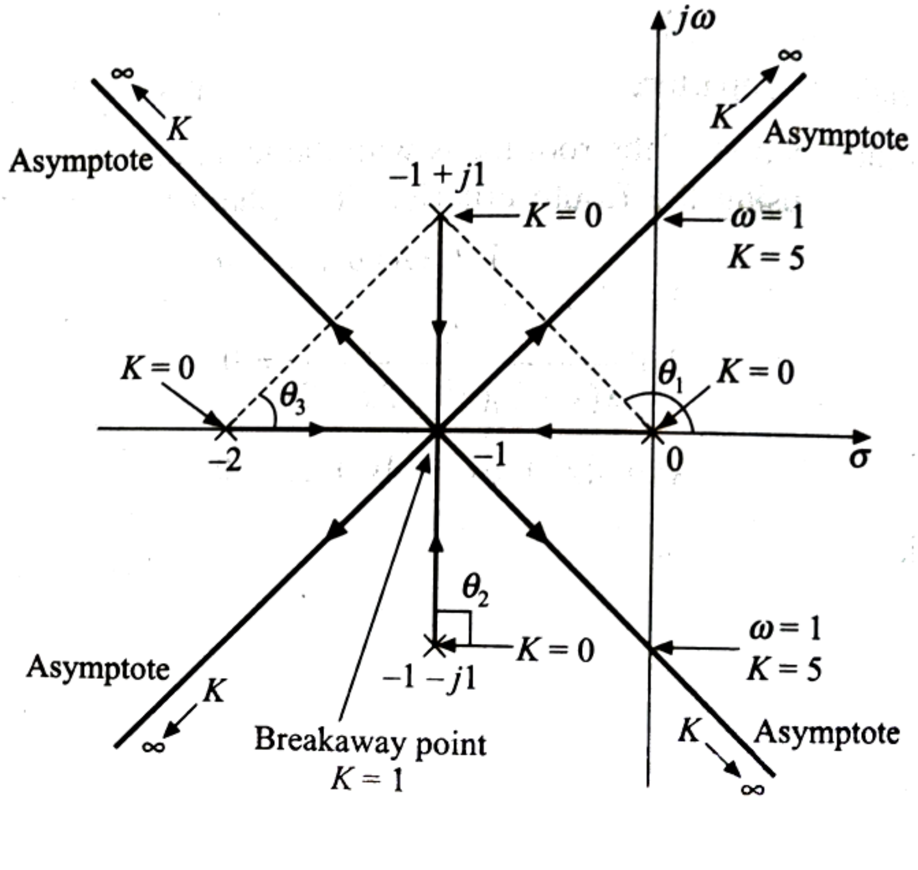
$$s^2 + 1 = 0$$

or

$$s = \pm j1$$

Therefore, the frequency of sustained oscillations is $\omega = 1$ rad/s.

The complete root locus is shown in Figure



4) Sketch the root locus for the system with the open-loop transfer function is

$$G(s)H(s) = \frac{K}{s(s+4)(s^2+4s+20)}$$

SOL:

$$s^2 + 4s + 20 = 0$$

$$\therefore\ s = \frac{-4 \pm \sqrt{16-80}}{2} = -2 \pm j4$$

For the given open-loop transfer function $G(s)H(s)$:

The open-loop poles are at $s = 0$, $s = -4$, $s = -2 + j4$, $s = -2 - j4$. Therefore, $n = 4$.

There are no open-loop zeros. Therefore, $m = 0$.

Hence the number of branches of root locus $= n = 4$ and the number of asymptotes $= n - m = 4 - 0 = 4$.

The complete root locus is drawn as shown in Figure , as per the rules given as follows:

1. Since the open-loop poles and zeros are symmetrical with respect to the real axis, the root locus will be symmetrical with respect to the real axis.
2. The four branches of the root locus originate at the open-loop poles $s = 0$, $s = -4$, $s = -2 + j4$, and $s = -2 - j4$, where $K = 0$ and terminate at the open-loop zeros at infinity, where $K = \infty$.
3. **Since there are no finite zeros, all the four branches of the root locus go to the zeros at infinity along straight line asymptotes, whose angles with the real axis are given by**

$$\theta_q = \frac{(2q+1)\pi}{n-m}, \quad q = 0, 1, 2, 3$$

$$\therefore \quad \theta_0 = \frac{\pi}{4}, \quad \theta_1 = \frac{3\pi}{4}, \quad \theta_2 = \frac{5\pi}{4}, \quad \theta_3 = \frac{7\pi}{4}$$

4. **The point of intersection of the asymptotes on the real axis (centroid) is given by**

$$-\sigma = \frac{\text{sum of real parts of poles} - \text{sum of real parts of zeros}}{\text{number of poles} - \text{number of zeros}} = \frac{(0-4-2-2)-(0)}{4-0} = -2$$

5. **The root locus exists on the real axis from $s = 0$ to $s = -4$.**

6. **The break points are given by the solution of $\frac{dK}{ds} = 0$.**

$$|G(s)H(s)| = \left|\frac{K}{s(s+4)(s^2+4s+20)}\right| = 1$$

$$\therefore \quad K = s(s+4)(s^2+4s+20)$$

$$\frac{dK}{ds} = -(4s^3 + 24s^2 + 72s + 80) = 0$$

i.e.
$$s^3 + 6s^2 + 18s + 20 = 0$$

$$(s+2)(s^2+4s+10) = 0$$

i.e.
$$(s+2)(s+2+j2.45)(s+2-j2.45) = 0$$

There is one breakaway point on the real axis at $s = -2$, and there are two complex conjugate breakaway points at $s = -2 \pm j2.45$.
The break angles at these break points are

$$\pm\frac{\pi}{r} = \pm\frac{180°}{2} = \pm 90°$$

7. The angle of departure from the complex pole $s = -2 + j4$ is given by

$$\theta_d = \pm(2q + 1)\pi + \phi$$

where $\phi = -(\theta_1 + \theta_2 + \theta_3) = -(117° + 90° + 63°) = -270°$

∴ $\theta_d = 180° - 270° = -90°$

Because of symmetry, the angle of departure from the complex pole at $s = -2 - j4$ is +90°.

8. The point of intersection of the root locus with the imaginary axis and the marginal value of K can be determined by use of the Routh criterion. The characteristic equation of the system is

$$1 + \frac{K}{s(s+4)(s^2+4s+20)} = 0$$

i.e $s(s + 4)(s^2 + 4s + 20) + K = 0$

i.e $s^4 + 8s^3 + 36s^2 + 80s + K = 0$

The Routh table is as follows:

s^4	1	36	K
s^3	8	80	
s^3	1	10	
s^2	26	K	
s^1	$\frac{260-K}{26}$		
s^0	K		

For stability, all the elements in the first column of the Routh array must be positive. Therefore,

$$K > 0$$

and $$\frac{260-K}{26} > 0$$

i.e. $$K < 260$$

So the range of values of K for stability is $0 < K < 260$. The marginal value of K for stability is $K_m = 260$. For $K = 260$, the roots lie on the imaginary axis.

The frequency of sustained oscillations is obtained by the solution of the auxiliary equation i.e. $A(s) = 26s^2 + K = 0$

i.e. $26s^2 + K_m = 0$

$\therefore$ $26s^2 + 260 = 0$

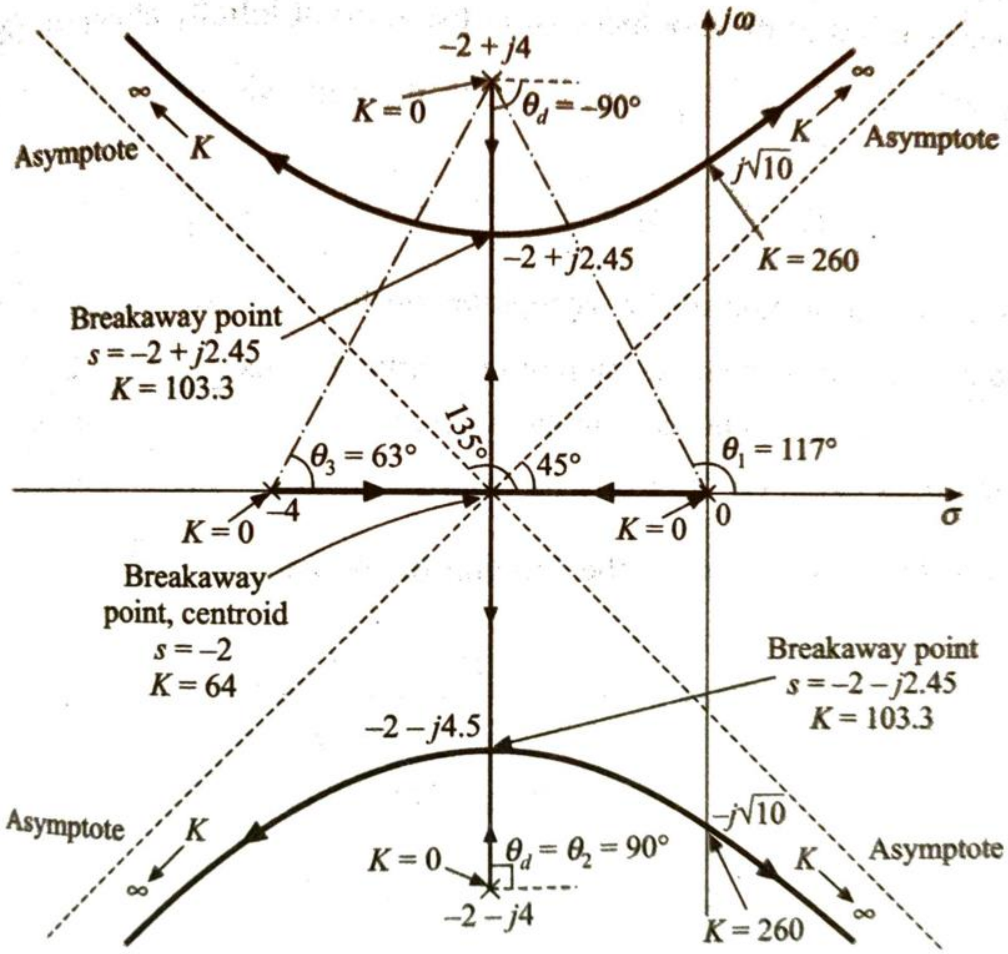
or $s^2 = -10$

$\therefore$ $s = \pm j\sqrt{10}$

$\omega = \sqrt{10}$ rad/s

Thus, for the root locus plot shown in Figure the branches intersect the $j\omega$-axis at $s = \pm j\sqrt{10}$ and the value of K corresponding to these roots is 260.

The complete root locus is drawn as shown in Figure



5) Sketch the root locus for the system with

$$G(s)H(s) = \frac{K(s^2 + 2s + 10)}{s^2(s + 2)}$$

SOL:

For the given open-loop transfer function $G(s)H(s)$:

The open-loop poles are at $s = 0$, $s = 0$, and $s = -2$. Therefore, $n = 3$.

The open-loop zeros are at $s = \dfrac{-2 \pm \sqrt{4 - 40}}{2} = -1 \pm j3$. Therefore, $m = 2$.

So the number of branches of root locus $= n = 3$ and the number of asymptotes $= n - m = 3 - 2 = 1$.

The complete root locus is drawn as shown in Figure as per the rules given as follows:

1. Since the open-loop poles and zeros are symmetrical with respect to the real axis, the root locus will be symmetrical with respect to the real axis.
2. **The three branches of the root locus start at the open-loop poles** $s = 0$, $s = 0$ and $s = -2$, **where** $K = 0$ **and terminate at the open-loop zeros** $s = -1 + j3$, $s = -1 - j3$ **and** $s = \infty$, **where** $K = \infty$.
3. **One branch of the root locus goes to the zero at infinity** along an asymptote making an **angle of** $\theta_q = \frac{(2q+1)\pi}{n-m}, q = 0,$ i.e. $\theta_0 = \pi$.
4. **The point of intersection of the asymptotes** on the real axis (centroid) is given by

$$-\sigma = \frac{\text{sum of real parts of poles} - \text{sum of real parts of zeros}}{\text{number of poles} - \text{number of zeros}} = \frac{(-2)-(-1-1)}{3-2} = 0$$

5. The breakaway point is at the origin itself. The break angles at $s = 0$ are

$$\pm\frac{\pi}{r} = \pm\frac{180^\circ}{2} = \pm 90^\circ$$

6. The root locus exists on the real axis to the left of $s = -2$.
7. The angle of arrival at the complex zero $-1 + j3$ is given by $\theta_a = \pm(2q+1)\pi - \phi$ where, from the graph

$$\phi = \theta_3 - (\theta_1 + \theta_2 + \theta_4) = 90^\circ - (108.4^\circ + 108.4^\circ + 71.6^\circ) = -198.4^\circ$$

$$\therefore \quad \theta_a = -180^\circ - (-198.4^\circ) = 18.4^\circ$$

The angle of arrival at the complex zero at $s = -1 - j3$ is $\theta_a = -18.4^\circ$.

8. The point of intersection of the root locus with the imaginary axis, and the critical value of K are obtained using Routh criterion. The characteristic equation is

$$1 + G(s)H(s) = 0 = 1 + \frac{K(s^2+2s+10)}{s^2(s+2)}$$

i.e. $s^3 + (2 + K)s^2 + 2Ks + 10K = 0$

The Routh table is as follows:

s^3	1	$2K$
s^2	$2 + K$	$10K$
s^1	$\frac{2K^2 + 4K - 10K}{(2+K)}$	
s^0	$10K$	

For stability all the elements in the first column of the Routh array must be positive Therefore,

$$10K > 0$$

i.e. $K > 0$

$$2 + K > 0$$

i.e $K > -2$

$$2K^2 - 6K > 0$$

i.e. $K > 3$

So the range of values of K for stability is $3 < K < \infty$. The marginal value of K for stability is $K_m = 3$.

The frequency of oscillations is given by the solution of the auxiliary equation

$$(2 + K)s^2 + 10K = 0$$

i.e. $(2 + K_m)s^2 + 10K_m = 0$

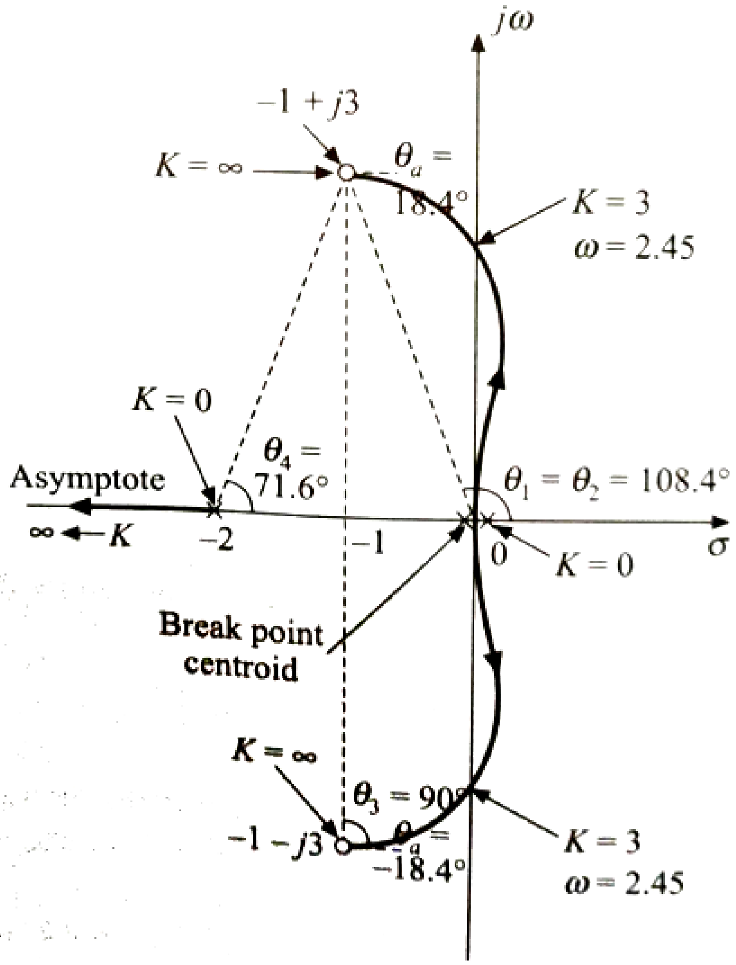
$$(2 + 3)s^2 + 10 \times 3 = 0$$

$$s^2 = -30/5 = -6$$

$$s = \pm j2.45$$

Therefore, the frequency of sustained oscillations is $\omega = 2.45$ rad/s.

The complete root locus is shown in Figure



6) The open loop transfer function of a system is given by

$$G(s)H(s) = \frac{K(s+12)}{s^2(s+20)}$$

Sketch the root locus for the system.

SOL:

Step 1 : Plot the poles and zero

Poles are at $s_1 = 0, s_2 = 0, s_3 = -20$

zero is at $s_4 = -12$

Step 2 : The segment between $s = -20$ and $s = -12$ is the part of the root locus.

Step 3 : Centroid of asymptotes

$$\sigma_A = \frac{\text{sum of poles} - \text{sum of zeros}}{P-Z} = \frac{0+0-20+12}{3-1} = -4$$

Step 4 : Angle of asymptotes

$$\phi = \frac{2K+1}{P-Z}180°$$

$K=0 \qquad \phi_1 = 90°$

$K=1 \qquad \phi_2 = 270°$

Step 5 : Breakaway point The characteristic eqn

$$1+\frac{K(s+12)}{s^2(s+20)} = 0$$

or,

$$K = -\frac{(s^3+20s^2)}{s+12}$$

$$\frac{dK}{ds} = -\left[\frac{(s+12)(3s^2+40s)-(s^3+20s^2)}{(s+12)^2}\right] = 0$$

or, $\quad s^3 + 28s^2 + 240\,s = 0$

$s\,(s^2 + 28s + 240) = 0$

we get $\quad s = 0, -14 \pm j\,6.63$

Breakaway point $s = 0$, points $-14 \pm j\,6.63$ are neither breakaway point nor breakin point, because the corresponding gain values K becomes complex quantities.

Step 6 : Point of intersection of rod locii with imaginary axis.

The characteristic eq^n.

$$s^3 + 20\,s^2 + Ks + 12\,K = 0$$

Put $s = j\omega$

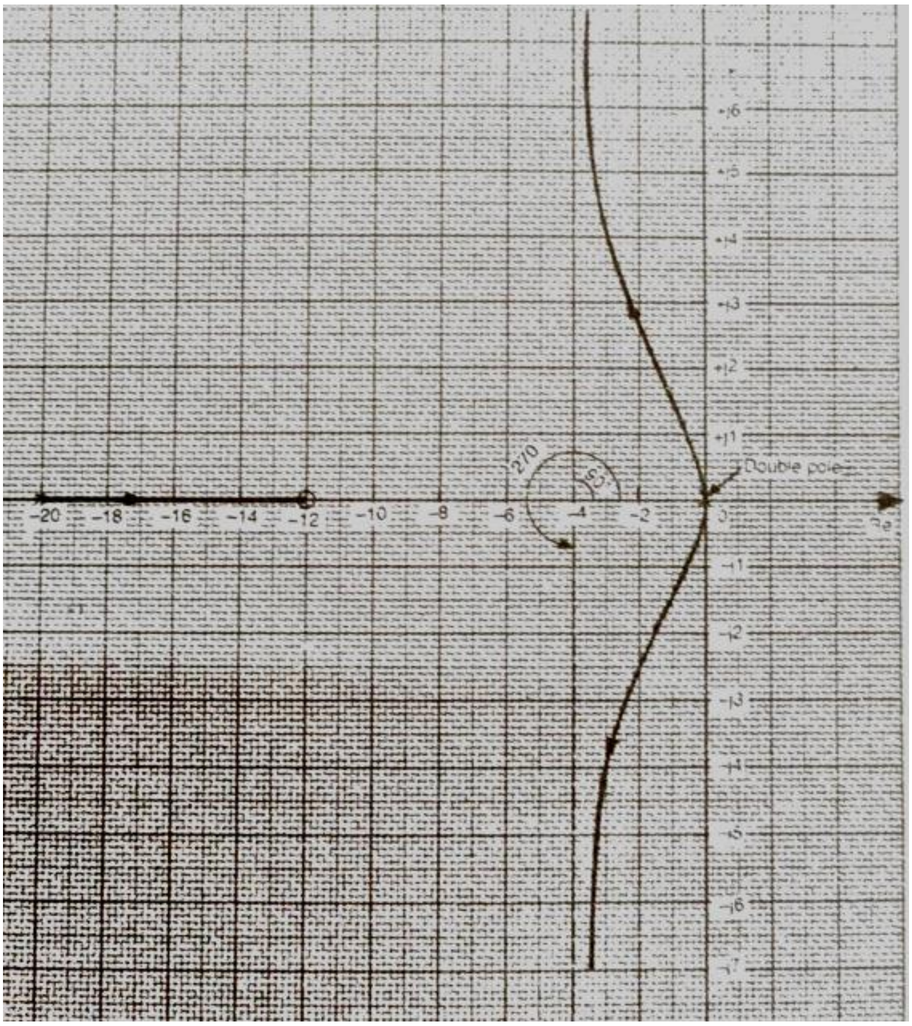
$$(j\omega)^3 + 20\,(j\omega)^2 + K\,(j\omega) + 12\,K = 0$$

$$(12\,K - 20\omega^2) + j\omega\,(K - \omega^2) = 0$$

if $\quad \omega = 0, \; K = 0$

because of double pole at the origin, the root locus is tangent to the imaginary axis at $\omega = 0$.

The root locus is shown in fig.



7) Sketch the root locus of the system whose characteristic equation is given by $S^4 + 6s^3 + 8s^2 + Ks + K = 0$.

SOL:

Expressing the given characteristic equation in the form

$$1 + G(s)\,H(s) = 0$$

$$1 + \frac{K(s+1)}{s^4 + 6s^3 + 8s^2} = 0$$

$$\therefore\ G(s)\,H(s) = \frac{K(s+1)}{s^2(s+2)(s+4)}$$

Step 1

Open loop zeros : – 1

Poles : 0, 0, – 2, – 4

Step 2

There are 4 root locus branches starting from the open loop poles and one of them terminates on open loop zero at *s* = – 1. The other three branches go to zeros at infinity.

Step 3

Angles of asymptotes

Since $n - m = 4 - 1 = 3$

$\phi = 60^0, 180^0, -60^0$

Step 4

Centroid

$$\sigma_a = \frac{-2-4+1}{3} = -\frac{5}{3}$$

Step 5

Root locus on real axis lies between – 1 and – 2, and – 4 to – ∞

Step 6

Break away point

The break away point is at s = 0 only.

Step 7

As there are no complex poles or zeros angle of arrival or departure need not be calculated.

Step 8

$j\omega$-axis crossing.

From the characteristic equation, Routh table is constructed.

Routh Table

s^4	1	8	K
s^3	6	K	
s^2	$\frac{48-K}{6}$	K	
s^1	$\dfrac{\frac{48K-K^2}{6}-6K}{\frac{48-K}{6}}$	0	
s^0	K		

A row of zeros is obtained when,

$$48\,K - K^2 - 36K^2 = 0$$

$$K = 0 \text{ or } K = \frac{48}{37}$$

K = 0 gives s = 0 as the cross over point.

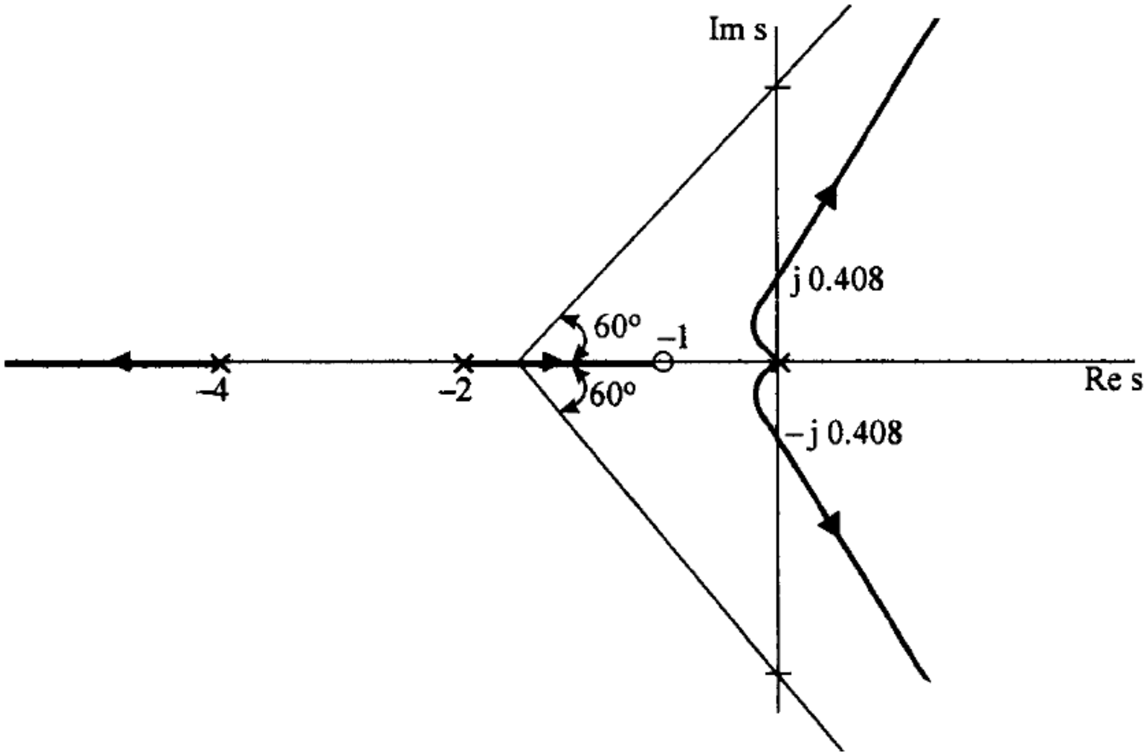
For $K = \frac{48}{37}$ forming auxiliary equation using s^2 row,

$$\frac{48-\frac{48}{37}}{6}s^2 + \frac{48}{37} = 0$$

$$7.784\,s^2 + 1.297 = 0$$

$$s = \pm j\,0.408$$

The complete root locus is sketched in Fig.


Im s
j 0.408
60°
−1
60°
−4
−2
Re s
−j 0.408